Kramers Wannier Duality

The Kramers Wannier duality is of central importance in the study of the 2D statmech models. We can use it to calculate the critical temperature of the 2D Ising and Potts model on some planar lattices, and also to obtain dual representations of spin-spin correlation functions.

1 Square lattice

First we will discuss the KW duality in the simplest case where it was first understood: the square lattice, where all bonds have the same strength J. Consider the Ising model with Hamiltonian given by

$$H(\{\sigma_i\}) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j, \tag{1}$$

with the sum $\langle ij \rangle$ over nearest neighbors on the square lattice in 2 dimensions. The Ising spins are binary degrees of freedom, each of which can assume the values $\sigma_i \in \{1, -1\}$. In order to calculate the partition function we wish to determine

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H(\{\sigma_i\})} \tag{2}$$

where β is inverse temperature. It will be convenient to define $K = \beta J$ in the following calculations.

1.1 High temperature expansion

The exact expression for the partition function is

$$Z(K) = \sum_{\sigma} \prod_{\langle ij \rangle} e^{K \sigma_i \sigma_j}. \tag{3}$$

We have explicitly written out the dependence on K since it is this functional dependence of Z that is important for calculating thermodynamic quantities. The important fact here is that for ± 1 spins, we have the identity $e^{x \sigma_i \sigma_j} = \cosh(x)(1 - \sigma_i \sigma_j \tanh x)$, since

$$1 \pm \tanh x = \frac{\cosh x \pm \sinh x}{\cosh x} = \frac{e^{\pm x}}{\cosh x}. \tag{4}$$

Defining N as the number of spins in the lattice, so that $2N$ is the number of bonds, we can exactly rewrite the partition function as

$$Z(K) = \cosh^{2N} K \sum_{\sigma} \prod_{\langle ij \rangle} (1 + \tanh K \sigma_i \sigma_j). \tag{5}$$

We see that if we expand the product along all of the $2N$ bonds, we will get 2^{2N} terms, each one of which corresponds to a subset of bonds in the lattice contributing a factor of $\tanh K$, and the rest of the bonds contributing a factor of 1. The only subsets of bonds which contribute to the partition function have an even number of bonds connected to each spin and therefore trace out a set of closed polygons, since each of the σ's must appear either 0, 2 or 4 times so as not to disappear under averaging. Each term corresponding to a configuration of closed polygons will, under summation over all spins, be multiplied by a factor of 2^N. Therefore our partition function assumes the form

$$Z(K) = 2^N \cosh^{2N} K \sum_{C \in \mathcal{CP}} \tanh^{|C|} K \tag{6}$$

where $|C|$ is the number of bonds in the closed polygon configuration C. Note that this expression for the partition function is still exact. We can truncate the expression and get a good approximation if $\tanh K$ is small (true at high temperature, since $K = J/T$) and realize that terms containing fewer bonds (therefore enclosing less area) contribute more to the partition function. With J positive (as in a ferromagnet) the loops all have the same sign: otherwise they have alternating sign depending on their length, but this expression for the partition function is exact in all cases, and is known as a *high temperature expansion* for aforementioned reasons.

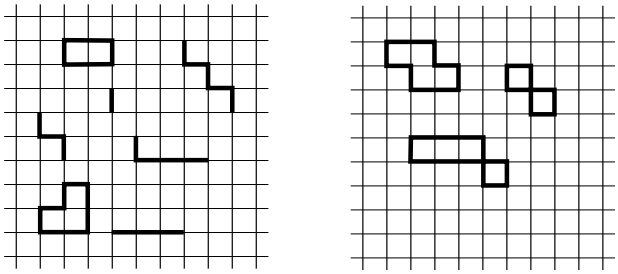

Figure 1: On the left: a picture of a diagram which does not contribute to the high temperature expansion of the partition function, because it averages to zero due to the presence of spins on which only one edge is incident. On the right: a term in the high temperature expansion corresponding to a configuration of closed polygons. Darkened edges represent factors of $\tanh K$. Here the spins live on the intersections of the grid lines, however the picture looks the same for the low temperature expansion, in which case spins live at the centers of plaquettes and the grid lines denote the dual lattice, with darkened edges contributing a factor of e^{-2K}.

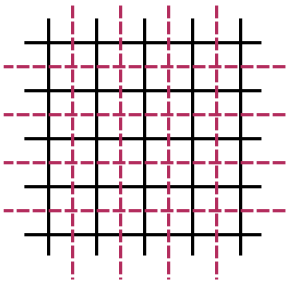

Figure 2: The primal lattice is shown in black (with spins sitting at intersections between black lines. The dual lattice is shown in dashed red lines, where spins on the dual lattice sit at intersections of red lines, and therefore at the centers of plaquettes on the primal lattice. Therefore closed polygons on the dual lattice encircle clusters of spins on the primal lattice and vice versa.

1.2 Low temperature expansion

At low temperature, the partition function is dominated by the spin configurations with all spins aligned or anti-aligned, since these are the lowest-energy states and contribute the most to the partition function with $\beta \to \infty$. The leading order corrections to the partition function then correspond to spin configurations with islands of spins flipped. A spin configuration with total island perimeter $|C|$ around clusters of overturned spins will contribute to the partition function a term of form $e^{2K(N-|C|)}$. Each spin configuration is characterized by a set of closed polygons on the *dual lattice*, since we can outline the droplets of overturned spins on the original (primal) lattice with closed polygons on the dual lattice. Therefore, once again, the partition function can be written as a sum over configurations of closed polygons.

$$Z(K) = 2 \sum_{C \in \mathcal{CP}} e^{2K(N-|C|)} = 2e^{2KN} \sum_{C \in \mathcal{CP}} e^{-2K|C|} \tag{7}$$

where $|C|$ is the number of bonds in polygon configuration C and N is the number of bonds in the lattice. Again, this expression for the partition function is exact if we truly sum over all sets of closed polygons on the dual lattice. At low temperatures the contribution from polygon configurations with large $|C|$ is small and can be neglected, so this expression for the partition function is called the *low temperature expansion*. We now have two exact expressions for the partition function: one in which we have factored out the infinite temperature partition function and kept a sum over closed polygons and the other in which we have factored out the zero temperature partition function and kept sum over closed polygons (but with a different function weighting each polygon configuration according to the number of bonds it contains).

1.3 Duality

We can now state the nontrivial exact equality

$$Z(K) = 2^N \cosh^{2N} K \sum_{C \in \mathcal{CP}} \tanh^{|C|} K = 2e^{2KN} \sum_{C \in \mathcal{CP}} e^{-2K|C|}. \tag{8}$$

The fact that both expressions for the partition function can be written in terms of sums over closed polygons is highly suggestive of a relationship between the two. In particular, the sums over the loops become the same if we consider the

partition function at two different values of the coupling, which we will denote K and K^* respectively, as long as these two values satisfy

$$\tanh K^* = e^{-2K}. \tag{9}$$

Note that the function relating K to K^* (i.e. $K = \operatorname{arctanh}[e^{-2K}]$) is an involution, i.e. it returns the original value when applied twice. This is a natural requirement for a duality relation. Then we have

$$Z(K) = 2e^{2KN} \sum_{C \in \mathcal{CP}} e^{-2K|C|} = 2 \tanh^{-N} K^* \sum_{C \in \mathcal{CP}} \tanh^{|C|} K^* \tag{10}$$

$$= \frac{2 \tanh^{-N} K^*}{2^N \cosh^{2N} K^*} Z(K^*) \tag{11}$$

$$= \frac{2}{\sinh^N (2K^*)} Z(K^*) \tag{12}$$

We have therefore derived a nontrivial relationship between the partition function of the 2D Ising model at two different couplings (and therefore two different temperatures) K and K^*. If we know the partition function at some K^*, we can work out what it must be at some other K which satisfies $\tanh K^* = e^{-2K}$, i.e. $K = -\frac{1}{2} \log \tanh K^*$. This is a map between partition functions at low temperatures and high temperatures, or vice versa.

We can see that when $\sinh 2K^* = 1$, we have that $Z(K) = \frac{1}{2} Z(K^*)$ and so the free energies per spin are identical in the thermodynamic limit. This indicates that there is a fixed point of our map which is *self dual*: in fact the condition $\sinh 2K^* = 1$ is equivalent to $\tanh K^* = e^{-2K^*}$, and so the map $K = -\frac{1}{2} \log \tanh K^*$ becomes the identity map $K = K^*$. This self dual point is exactly the critical point of the Ising model. We can solve the equation $\tanh K^* = e^{-2K^*}$ to find that the solution for the critical coupling is

$$K^* = \frac{1}{2} \log(1 + \sqrt{2}), \tag{13}$$

implying a critical temperature of

$$T_c = \frac{2J}{\log(1 + \sqrt{2})}. \tag{14}$$

To reiterate, the special property of this self-dual point is that the duality map that exchanges high and low temperature becomes the identity map, implying that this critical point is neither in the high nor the low temperature phase.

1.4 Gauge transformations

The Ising model on the square lattice has crucial symmetry properties that are very useful. In particularly it displays a *gauge symmetry* such that the partition function of the model is invariant under flipping the signs of bonds in the shape of a plus sign, which correspond to flipping the signs of a closed loop of bonds on the dual lattice (since the bonds in the dual lattice have the values of the bonds they cross in the primal lattice and vice versa).

This gauge symmetry is a local version of the overall symmetry (only on bipartite lattices) in the partition function when the sign of all the bonds is flipped. This symmetry arises because for each pair of spin configurations related by a global spin flip, if we flip all the bonds, we have another pair of spin configurations with the same energy: the original configuration with the two sublattices flipped with respect to one another. Therefore on a square lattice the ferromagnet and antiferromagnet therefore have the same partition function.

On nonbipartite lattices, which can't be split into two sublattices which only couple to one another, the partition function is not invariant under a global flip of all the bonds, since there is no such correspondence between pairs of spin configurations with and without global bond flip. There is similarly no analogous version of a gauge symmetry.

1.5 Anisotropic square lattice

We can generalize this duality relation to the case where horizontal and vertical bonds have different strengths J_x and J_y respectively. This will allow us to see why the critical point of the transverse field Ising model [TFIM] (which is an anisotropic limit of the 2D classical Ising model) is the same as that of the 2D classical model [1].

In the appropriate anisotropic limit of the 2D classical model, namely when $K_x \sim J$ and $e^{-2K_y} \sim h$ for horizontal and vertical couplings K_x and K_y and quantum exchange interaction J and transverse field h, the free energy of the classical model at infinite size in the vertical direction agrees with the free energy of the quantum chain at zero temperature.

One can extend the argument for the location of the self-dual point above to the case where the couplings in the horizontal and vertical directions are not the same; in this case the location of the critical point is fixed by

$$\sinh 2K_x \sinh 2K_y = 1. \tag{15}$$

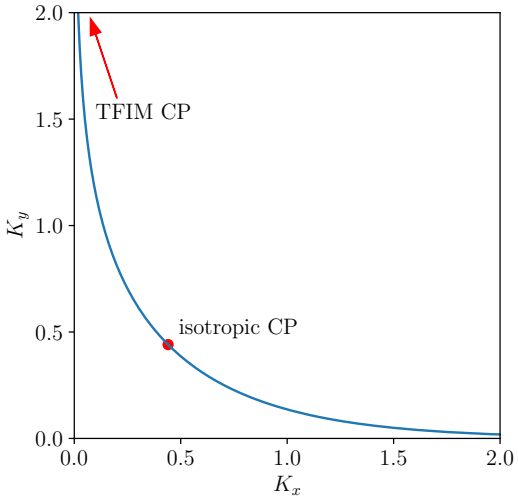

Figure 3: The critical line in the space of K_x and K_y for the anisotropic Ising model on a square lattice. The isotropic critical point and the critical point of the transverse field Ising model both lie on the the same critical line.

We can see that as $K_y \to \infty$ this implies that $K_x \to 0$ as $K_x \approx e^{-2K_y}$. This is exactly the critical point of the quantum transverse field Ising model which occurs at $J = h$ and therefore $K_x = e^{-2K_y}$. Therefore see that the line of critical points in (K_x, K_y) space connects both the TFIM critical point and the 2D classical Ising critical point. This is the heuristic reason why the phase transitions in both models are in the same universality class.

2 Triangular lattice

We saw that duality relates the partition function on the lattice to a partition function on the dual lattice (defined by placing nodes in the middles of plaquettes and edges crossing those in the primal lattice) at the "opposite" temperature. The nice thing about the square lattice is that the lattice is self-dual, so this boils down to a relationship between the partition function on the square lattice at low and high temperature. For a non-square lattice e.g. the trangular lattice, duality will relate the partition function on the triangular lattice at high temperature to that of the *honeycomb* lattice at low temperature, since the honeycomb and triangular lattices are dual to each other. However in order to find the critical temperature of the Ising model on either triangular or honeycomb lattices, it is necessary to relate the high and low temperature partition functions on the *same lattice*. This can be done with the use of the *star triangle* identity.

2.1 Duality relation

We start by writing an expression for the partition function on a triangular lattice as a high temperature expansion, which, for N spins and therefore $3N$ bonds, is given by

$$Z_{\text{triang}}(N, K) = 2^N \cosh^{3N} K \sum_{C \in \mathcal{CP}} \tanh^{|C|} K. \tag{16}$$

Note that we can also write the partition function of an Ising model on a honeycomb lattice as a sum over the same set of loops, with a low temperature expansion. However the honeycomb lattice dual to a triangular lattice with N spins contains $2N$ spins (and $3N$ bonds)—therefore we need $2N$ spins in the honeycomb lattice to get the same partition function:

$$Z_{\text{hex}}(2N, K) = 2e^{3NK} \sum_{C \in \mathcal{CP}} e^{-2K|C|}. \tag{17}$$

Therefore if we choose $e^{-2K} = \tanh K^*$ then there is a relation between the honeycomb and triangular lattice partition functions: namely,

$$Z_{\text{hex}}(2N, K) = \frac{2}{2^{-N/2} \sinh^{3N/2}(2K^*)} Z_{\text{triang}}(N, K^*). \tag{18}$$

This establishes a duality between the partition function of the honeycomb lattice and triangular lattice at dual temperatures. As in the case of the square lattice, when $\sinh(2K^*) = 1$ the free energies are equal. However, in contrast to the square lattice case, this does not imply a critical point because the equivalence of free energies is for models on different lattices. In order to find the critical point we will derive a relation between the partition function at the same temperature on both the triangular and hexagon lattice; this relation is known as the *star-triangle transformation*.

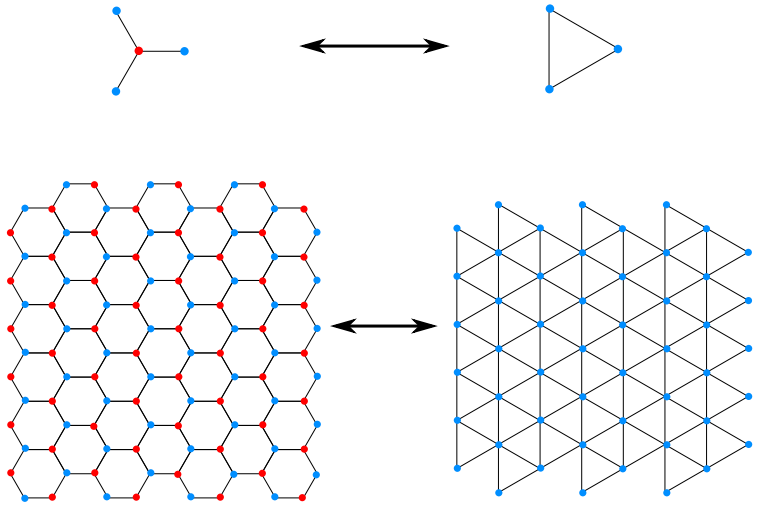

Figure 4: Above, we show the correspondence between two arrangements of spins in the partition function, where the red spin is summed over to give effective pairwise interactions between the remaining blue spins. If this procedure is generalized on the honeycomb lattice, we can relate the partition function of the Ising model on a honeycomb lattice with that on a triangular lattice with half the number of spins (and the same number of bonds).

2.2 Star-triangle transformation

The idea is to consider the Ising partition function on a honeycomb lattice, and imagine summing over the spins on one of the sublattices (since the honeycomb lattice is bipartite). Then one can write this remaining partition function in terms of an effective Hamiltonian for pairwise interactions among the remaining spins on the other sublattice, which is a triangular lattice.

Here we will discuss the isotropic example which is the simplest case in which all the bonds on both the honeycomb and triangular lattices are the same. However in general this transformation can be modified to include cases where there are three types of bonds on each of the triangular and honeycomb lattices: a more complete discussion is in Baxter's book [2].

The key point is that the partition function on the honeycomb lattice with coupling L is a sum over spins of the expression

$$\prod_{i,j,k,l} e^{L\sigma_l(\sigma_i+\sigma_j+\sigma_k)} \tag{19}$$

where the l index is over spins on one of the sublattices, and the i, j, k denote the spins on the other sublattice which neighbor spin l. We can sum over the sublattice inhabited by the $\{\sigma_l\}$ to obtain an effective interaction between the σ_i, σ_j and σ_k, as shown in Figure 4 where we sum over the red sublattice to get an effective model on the triangular lattice. Recall that the triangular lattice partition function is a sum over spins of

$$\prod_{i,j,k} e^{K(\sigma_i\sigma_j+\sigma_i\sigma_k+\sigma_j\sigma_k)} \tag{20}$$

where spins i, j, k live on the vertices of a triangular plaquette. To find the value of K for which $Z_{\text{triang}}(N, K)$ is related to $Z_{\text{hex}}(2N, L)$, we conduct the sum over the sublattice in the honeycomb containing the $\{\sigma_l\}$, and then require that the remaining terms in Z_{hex} match term-by-term with the terms in Z_{triang}, both of which contain only spins living on a single triangular plaquette. Therefore we can write that

$$2\cosh L(\sigma_i+\sigma_j+\sigma_k) = R\exp K(\sigma_j\sigma_k+\sigma_i\sigma_j+\sigma_i\sigma_k) \tag{21}$$

where R and K depend on L in a way that is fixed by equality of the partition functions on the triangular and honeycomb lattices. In order to ensure validity, we need

$$2\cosh 3L = Re^{3K} \tag{22}$$

$$2\cosh L = Re^{-K}, \tag{23}$$

which come from considering all possible values of σ_i, σ_j and σ_k, i.e. all aligned or one out of three anti-aligned, yielding two conditions on K and R.

Therefore we have $R^2 = 4\cosh L \cosh 3L e^{-2K}$ where L and K are related by $\cosh 3L = e^{4K}\cosh L$. We can solve this system of equations to find that

$$R^2 = 3e^{2K} + e^{6K}. \tag{24}$$

We now have a relation between the partition function on the isotropic triangular and honeycomb lattices, which we can use to find the critical point of the Ising model on both the triangular and honeycomb lattices. Namely, we can say that

$$Z_{\text{hex}}(2N, L) = R^N Z_{\text{triang}}(N, K) \quad \text{with } e^{4K} = \frac{\cosh 3L}{\cosh L}. \tag{25}$$

Note that if K is large then L is large: this is therefore a relation between triangular and honeycomb partition functions at similar temperatures. We can combine this relation with the duality relation (relating low and high temperatures) from earlier to say that

$$Z_{\text{hex}}(2N, L) = \frac{2}{2^{-N/2}\sinh^{3N/2}(2L^*)} Z_{\text{triang}}(N, L^*) = R^N Z_{\text{triang}}(N, K) \quad \text{with } \tanh L^* = e^{-2L}. \tag{26}$$

Therefore the condition for criticality in the triangular lattice is

$$\frac{1}{2}\sinh^3(2L^*)R^2 = 1. \tag{27}$$

Combined with the definition of R, and the relations $\tanh L^* = e^{-2L}$ and $\cosh 3L = e^{4K}\cosh L$, one can check that the critical K that satisfies all these conditions is

$$\sinh 2K_c = \frac{1}{\sqrt{3}}. \tag{28}$$

To find the critical coupling for the honeycomb lattice, we again combine the two relations to see that

$$2Z_{\text{triang}}(N, K) = 2^{-N/2}\sinh^{3N/2}(2K) Z_{\text{hex}}(2N, K^*) = \frac{2}{R^N} Z_{\text{hex}}(2N, L). \tag{29}$$

Now the condition for criticality in the honeycomb lattice is

$$\frac{1}{2}\sinh^3(2K)R^2 = 1, \tag{30}$$

along with the definition of R and $\tanh K^* = e^{-2K}$ as well as $\cosh 3L = e^{4K}\cosh L$. One can check that L_c satisfying $\sinh 2L_c = \sqrt{3}$ satisfies these conditions.

2.3 Summary

Using the star-triangle transformation in conjunction with the duality transformation, we were able to show that the critical coupling is $K_c = \frac{1}{2}\operatorname{arcsinh}\sqrt{3}$ in the honeycomb lattice and $L_c = \frac{1}{2}\operatorname{arcsinh}\frac{1}{\sqrt{3}}$ in the triangular lattice.

3 Disorder operator

We have seen that partition functions and therefore free energies above and below the critical temperature are related by duality. What about other objects, like spin-spin correlation functions? This motivates the introduction of *disorder operators*, whose correlation functions are the objects dual to spin-spin correlations. In particular, the correlation function of disorder operators is defined on the dual lattice as

$$\langle \mu_0 \mu_r \rangle_K = \frac{Z_\Gamma(K^*)}{Z(K^*)}, \tag{31}$$

where Z_Γ refers to the partition function of the system with a line of bonds pierced by a path on the dual lattice between 0 and r flipped in sign. For a ferromagnetic Ising model, this flipping will introduce two frustrated plaquettes at the endpoint of the string of flipped bonds, and will therefore increase the free energy such that the ratio of the partition functions is $\exp(-\Delta F[\Gamma]/T)$ where $\Delta F[\Gamma]$ is the increase in free energy caused by introducing the pair of frustrated plaquettes. In the ferromagnet, introducing a pair of frustrated plaquettes costs energy, and so the disorder correlation function will decay to zero as $|r - r'| \to \infty$. However in the paramagnetic phase, domain wall tension is zero and so the disorder operator asymptotes to 1 as $|r - r'| \to \infty$ — since it acquires a nonzero expectation in the disordered phase, it is called the *disorder* operator.

Kadanoff and Ceva [3] showed that the quantity $\langle \mu_0 \mu_r \rangle$ is exactly equal to the *spin* correlation function $\langle \sigma_0 \sigma_r \rangle$ for spins on the dual lattice, at the dual temperature. In other words, if we want to calculate the correlation function of two spins,

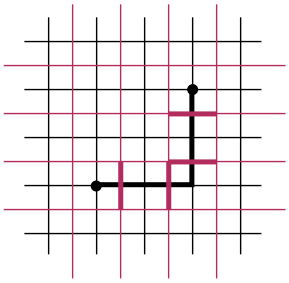

Figure 5: The primal lattice is shown in black and the dual lattice in red. The two spins whose correlation function we wish to compute are shown as black circles. If we pick a path between the spins on the primal lattice (bold black) and flip the signs of the bonds on the dual lattice that cross this path (the bold red bonds) then the ratio of this partition function at the dual temperature with the flipped bonds to the regular partition function at the dual temperature gives us the spin-spin correlation at the primal temperature.

we can calculate correlation function of disorder operators on the dual lattice at the dual temperature. This mapping makes it apparent that the disorder correlation function asymptotes to 1 in the paramagnet, since when the primal lattice is paramagnetic the dual lattice is ferromagnetic so the spin correlation function asymptotes to 1.

We can show that the Kramers Wannier dual of a correlation function of spins is a correlation function of disorder operators as follows: we first show that the correlation function between two spins for some coupling K can be written (up to factors of i) as a ratio of partition functions where the numerator has the couplings between the two spins transformed as $K \mapsto K + i\pi/2$ along a path between the two spins, and the denominator is the regular partition function. Note that *which* path we flip the bonds along does not matter, as a consequence of the gauge symmetry discussed above. We then have

$$Z_\Gamma(K) = \sum_{\{\sigma\}} \prod_{\langle ij\rangle \notin \Gamma} e^{K\sigma_i\sigma_j} \prod_{\langle ij\rangle \in \Gamma} e^{K\sigma_i\sigma_j} \underbrace{e^{i\frac{\pi}{2}\sigma_i\sigma_j}}_{=i\sigma_i\sigma_j} \tag{32}$$

where Γ is a path on the primal lattice connecting spins σ_0 and σ_r, and Z_Γ denotes the partition function with signs of the bonds along this path flipped. Therefore

$$\frac{Z_\Gamma(K)}{Z(K)} = i^{|\Gamma|}\langle\sigma_0\sigma_r\rangle_K \tag{33}$$

since all squared spins are 1 and so only the endpoints of Γ, at 0 and r, contribute meaningfully to the product in the expression for Z_Γ.

Now we note that making the transformation $K \mapsto K + i\pi/2$ on the primal lattice means that K^* on the dual lattice will change sign, since $K^* = \operatorname{arctanh}[e^{-2K}]$. Using the duality transformation $Z(K) = \frac{2}{\sinh^N(2K^*)} Z(K^*)$ generalized to the case where not all the bonds are the same strength, in which case it reads

$$Z(\{K_{ij}\}) = \frac{2Z(\{K^*_{ij}\})}{\prod_{\langle ij\rangle}\sqrt{\sinh 2K^*_{ij}}}, \tag{34}$$

we have

$$i^{|\Gamma|}\langle\sigma_0\sigma_r\rangle_K = \frac{\sinh^N(2K^*)}{\prod_{\langle ij\rangle\notin\Gamma}\sqrt{\sinh 2K^*_{ij}}\prod_{\langle ij\rangle\in\Gamma}\sqrt{-\sinh 2K^*_{ij}}} \frac{Z_\Gamma(K^*)}{Z(K^*)} = \frac{Z_\Gamma(K^*)}{(-i)^{|\Gamma|}Z(K^*)} \tag{35}$$

where we have taken the negative branch of the square root. Therefore we have shown that

$$\langle\sigma_0\sigma_r\rangle_K = \frac{Z_\Gamma(K^*)}{Z(K^*)}. \tag{36}$$

In other words, spin correlations at a particular temperature are equal to the ratio of partition functions at the dual temperature. These statements can be generalized for Ising models with arbitrary couplings, not just uniform couplings as we have seen here.

4 Inhomogeneous bond strengths

What about the case of heterogenous bonds in the lattice? Here the high and low temperature expansions generalize straightforwardly, and they give us two ways of computing the partition function. Now when we sum over loops, we care

not only about the length of the loop, but about all the bonds in the loop. If we denote the coupling between spin i and j as J_{ij} and $K_{ij} = \beta J_{ij}$ then for the high temperature expansion we have

$$Z(\{K_{ij}\}) = 2^{N/2} \left[\prod_{\langle ij \rangle} \cosh K_{ij} \right] \sum_{C \in \mathcal{CP}} \prod_{\langle kl \rangle \in C} \tanh K_{kl} \tag{37}$$

where the polygon configurations C live on the primal lattice.

The same procedure works for the low temperature expansion, where now the sum will be over $\exp(-2\sum_{\langle kl \rangle \in C} K_{kl})$. However, if we want to define couplings K^*_{ij} via $\tanh K^*_{ij} = e^{-2K_{ij}}$ such that $Z(\{K_{ij}\})$ and $Z(\{K^*_{ij}\})$ have the same expression for the sum over loops, then one of the sets of couplings must contain imaginary values, since the $\{K_{ij}\}$ can be of both signs. So the mapping between the two models related by duality is less straightforward. However, the high and low temperature expansions do give us two different ways to write down the partition function for arbitrary collections of bonds.

5 Potts duality

The KW duality can again be extended to consider 2D Potts models, of which we will be able to determine the critical temperature. The Potts Hamiltonian is given by

$$\beta H = -K \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}, \tag{38}$$

where the spins can take on one of q different values, and positive J assigns a negative energy to aligned spins. Note that $q = 2$ recapitulates the Ising model (up to an overall shift and rescaling in the energy since the Kronecker delta is 0 or 1 rather than -1 or 1). As in the Ising case, we can construct a high and low temperature expansion of the Potts partition function; on the square lattice this will be sufficient to determine the critical point since the primal and dual lattices are the same. There is some additional cleverness needed to write the high temperature expansion as a sum over closed polygons, since the Potts spins do not automatically lead to terms with a single spin summing to zero. The main result [4] is that the duality relation between K and K^* for which the partition functions are related is modified to

$$(e^K - 1)(e^{K^*} - 1) = q, \tag{39}$$

which means that there is a transition at $K_c = \log(1 + \sqrt{q})$. Note the factor of two difference between the result for the Ising model when $q = 2$; this is due to the fact that the energy difference between aligned and anti-aligned is 1 for Potts spins and 2 for Ising spins.

One can implement a generalization of the star-triangle relation for the q-state Potts model to get a relation between the partition functions on a triangular and honeycomb lattice and, combined with the duality above, this allows exact expressions for the transition temperature of the q-state Potts model on triangular and honeycomb lattices [5].

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