The Bethe Ansatz

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1 Introduction

The Bethe Ansatz has found widespread use in questions from condensed matter to stochastic processes to high energy physics. But what is it? At the most elementary level, it as a guess for the form of the eigenvectors of a matrix that in some cases allows us to diagonalize a matrix with certain properties, thereby yielding its complete spectrum. However the Bethe ansatz, after its initial invention by Hans Bethe in 1931, has taken on a life of its own and essentially underlies the field of integrable systems: in some sense "integrable" is synonymous with "Bethe ansatz solvable." In these notes we will outline the coordinate Bethe ansatz in the context that led to its creation, and deploy it in considerable detail. We will also discuss the highly nontrivial fact that the Bethe ansatz works at all, and what this means in terms of simplifying the problem we wish to solve. The problem we discuss below is that of finding the spectrum of the Heisenberg spin 1/2 XXX chain: a model of a one dimensional quantum magnet with nearest-neighbor isotropic spin interactions.

We do not assume any knowledge of the Bethe ansatz, but it will be helpful to be familiar with the quantum mechanics of spin: in particular the Pauli spin matrices, and the representation of quantum mechanical Hilbert space as a tensor product of local spaces. We should state at the outset that if one's goal is to understand the Bethe ansatz in general without an interest in quantum mechanics, there are similarly illuminating use cases for diagonalizing the Markov matrix of particular families of stochastic processes — we will discuss the latter in a separate set of notes. For now we will proceed with the quantum mechanical problem, which will soon be converted into a problem in linear algebra.

2 The Heisenberg spin chain

The model of interest here will be the Heisenberg spin 1/2 chain, which is a quantum model of magnetism in one dimension. Its Hamiltonian can be represented as a sum of local products of spin operators:

$$H = J \sum_{i=1}^{L} \mathbf{S}_i \cdot \mathbf{S}_{i+1}, \tag{1}$$

where the vector $\mathbf{S}_i$ denotes the spin operator (S_i^x, S_i^y, S_i^z) on site i, comprised of the spin operators in each of the spatial directions. We will use periodic boundary conditions so that $L+1$ is identified with 1. Hilbert space grows exponentially with the number of particles, and so H is fact a $2^L \times 2^L$ matrix, as we will see explicitly below. The spin operators on a particular site of the chain are given by

$$S_i^\alpha = \mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_{i-1} \otimes \sigma_\alpha \otimes \mathbb{1}_{i+1} \otimes \cdots \otimes \mathbb{1}_L, \tag{2}$$

for $\alpha = x, y, z$, with $\sigma_x, \sigma_y, \sigma_z$ representing the Pauli spin 1/2 matrices, which are 2×2 matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Here $\mathbb{1}$ is a 2×2 identity matrix, irrespective of its index, which simply labels the site on which it acts. The tensor product symbol $\otimes$ is a way of combining the degrees of freedom from each of the individual Hilbert spaces of the spins into one large Hilbert space for the whole system. Explicitly in a matrix representation, we have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}. \tag{4}$$

The sign of J in Equation 1 determines whether the chain is ferromagnetic or antiferromagnetic. Note the similarity between this model and the one dimensional classical Ising model. The difference is that here each spin has three components in the x, y and z directions respectively, and these are represented by non-commuting operators.

3 Constructing the Bethe ansatz

As discussed earlier, we want to find the eigenvalues and eigenvectors of the Hamiltonian defined in Equation 1. This will give us the eigenstates and corresponding energies of the magnet, and in principle will fully solve the system and its dynamics, allowing us to calculate whatever we want. Things will not quite turn out to be that nice, but we can still

make significant progress with the Bethe ansatz. The key will be to think about how the Hamiltonian operator changes a state of the system when applied to it. We know that a state of the system will be a 2^L-dimensional vector. Let us work in the z basis, in which case our basis vectors are states of the system described by the z component of the spin at each site, which can be 1 or -1, corresponding to $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ respectively. These states can be represented as a tensor product of spins $\uparrow$ and $\downarrow$ at each site, in which case we might use a shorthand like $|\uparrow\uparrow\downarrow\rangle$ for a chain of three spins, which corresponds to state vector

$$|\uparrow\uparrow\downarrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}. \tag{5}$$

We will try to form eigenvectors of our Hamiltonians are linear combinations of states of this form. To simplify notation, we will use the notation $|n_1, \ldots, n_p\rangle$ to denote a state with down spins at sites on the chain except for the sites $n_1, \ldots n_p$. Explicitly, these means that for $L = 3$, we have $|1\rangle = |\downarrow\uparrow\uparrow\rangle$, and $|1, 3\rangle = |\downarrow\uparrow\downarrow\rangle$, etc. Eventually we will want to take L to be very large. Then states with, e.g. one spin flipped relative to all the others will correspond to the high (or low) edge of the density of states for positive (or negative) J. We will think of these states as "excitations" of p spins away from our reference configuration of all spins up.

3.1 Single spin excitations

First, we would like to study, for arbitrary L, how we might construct an eigenfunction of H out of a linear combination of states with $p = 1$, i.e. a single spin pointing down while all others point up. Such a state can be notated as $|n\rangle$ where n is the index of the site at which the down-spin is located. The crucial observation here is that

$$H|n\rangle = (L-2)|n\rangle + 2[|n-1\rangle + |n+1\rangle - |n\rangle]. \tag{6}$$

How can we see this? Thinking of the Hamiltonian H as a sum of many local products of spin operators acting on sets of neighboring spins, as in Equation 1, most of the bonds ($L-2$ of them) will simply be acted on by the identity operator. The nontrivial contribution will come from the bonds (on either side of the nth site) that are not satisfied. By explicit computation with the 4 dimensional subspace of Hilbert space corresponding to two neighboring spins which are oppositely oriented, we can come to the conclusion above.

Now we see that the family of states indexed by n has the nice property that we can explicitly write down the operation of H on a single one of these states in terms of other states indexed by different n. This will allow us to solve for eigenstates of H as a linear combinations of the $|n\rangle$. We guess an eigenstate $|\Psi\rangle = \sum_n z^n |n\rangle$. Therefore the condition on $|\Psi\rangle$ in order for it to be an eigenstate of H with eigenvalue λ is

$$\lambda|\Psi\rangle = \sum_n \left[z^n(L-4) + 2z^{n-1} + 2z^{n+1}\right]|n\rangle. \tag{7}$$

Another way to notate this (which we will use hereafter) is if we define $\psi(n)$ to be the coefficient in front of $|n\rangle$. Then we can write

$$\lambda\psi(n) = (L-4)\psi(n) + 2\psi(n-1) + 2\psi(n+1). \tag{8}$$

We additionally have $\psi(n+L) = \psi(L)$ from periodic boundary conditions. Plugging in the ansatz $\psi(n) = z^n$ tells us that $\lambda = L - 4 + 2z^{-1} + 2z$ from the eigenvalue conditions, and $z^L = 1$ from periodicity. Therefore we find eigenvectors if $z = e^{i2\pi k/L}$, with corresponding eigenvalues $\lambda_k = L - 4 + 2z^{-1} + 2z = L + 4[\cos(2\pi k/L) - 1]$, for $k \in \{1, \ldots, L\}$.

We have therefore found L eigenvector-eigenvalue pairs for our Hamiltonian in terms of linear combinations of the $|n_1, \ldots n_p\rangle$ states with $p = 1$. However this is not the whole story — in particular there should be 2^L eigenvector-eigenvalue pairs in total. Below we will outline the procedure for finding the rest of these, which leads us to the Bethe ansatz.

3.2 Two-spin excitations

Here we will outline the case of $p = 2$: that is, we will try to form an eigenstate from a linear combination of states with two down-spins somewhere in the chain of up-spins. These states can be notated as $|n, m\rangle$ with $1 \le n < m \le L$. Following our approach for $p = 1$, the crucial observation is that

$$H|n, m\rangle = (L-8)|n, m\rangle + 2[|n+1, m\rangle + |n-1, m\rangle + |n, m+1\rangle + |n, m-1\rangle] \tag{9}$$

$$H|n, n+1\rangle = (L-4)|n, n+1\rangle + 2[|n-1, n+1\rangle + |n, n+2\rangle]. \tag{10}$$

This first of these observations is a natural generalization of Equation 6 to the case $p = 2$, which assumes the two down-spins are spatially separated and therefore act just like isolated single-spin excitations. However we need to separately treat the case where these two spin excitations are neighbors, and that is where the second equation comes in. In this

case we have two down-spins next to each other in the chain of up-spins, and accordingly the number of unsatisfied bonds is two. By considering the local action of the successive terms in the Hamiltonian on our chain carefully, we can come up with the second equation above. We will refer to this as a "collision condition," since it treats the case where our two spin excitations "collide" in space. Rewriting these two conditions in our alternative notation gives

$$\lambda\psi(n,m) = (L-8)\psi(n,m) + 2\psi(n-1,m) + 2\psi(n+1,m) + 2\psi(n,m-1) + 2\psi(n,m+1) \tag{11}$$

$$\lambda\psi(n,n+1) = (L-4)\psi(n,n+1) + 2\psi(n-1,n+1) + 2\psi(n,n+2). \tag{12}$$

In addition, periodic boundary conditions give us $z_1^L z_2^L = 1$ and $\psi(n, L+1) = \psi(1,n)$. [Think about why this second statement is the correct boundary condition!]

Now we make the ansatz

$$|\Psi\rangle = \sum_{1\le n<m\le L} (Az_1^n z_2^m + Bz_2^n z_1^m)|n,m\rangle \implies \psi(n,m) = Az_1^n z_2^m + Bz_2^n z_1^m. \tag{13}$$

In addition to determining z_1 and z_2, we need to determine the ratio A/B (the overall amplitude of the eigenvector is a free parameter).

Plugging in this ansatz, we conclude that

$$\lambda = L - 8 + 2\left(z_1 + z_2 + \frac{1}{z_1} + \frac{1}{z_2}\right). \tag{14}$$

Then from the cancellation condition we get

$$\frac{A}{B} = -\frac{2z_1 - 1 - z_1 z_2}{2z_2 - 1 - z_1 z_2}. \tag{15}$$

The boundary conditions then give $z_1^L z_2^L = 1$ and

$$\frac{A}{B} = \frac{z_2 z_1^n - z_2^n z_1^{L+1}}{z_1^n z_2^{L+1} - z_1 z_2^n} = z_1^L, \tag{16}$$

where the second equality comes from using the fact that $z_1^L z_2^L = 1$. This allows us to eliminate the ratio A/B, and we obtain two equations that constrain z_1 and z_2:

$$z_1^L z_2^L = 1 \quad \text{and} \quad z_1^L = -\frac{z_1 z_2 - 2z_1 + 1}{z_1 z_2 - 2z_2 + 1}. \tag{17}$$

Note that if we want we can write these symmetrically for z_1 and z_2, namely

$$z_i^L = -\prod_{k=1}^{2} \frac{z_i z_k - 2z_i + 1}{z_i z_k - 2z_k + 1}. \tag{18}$$

Solutions to these equations will allow us to determine the eigenvalues and eigenvectors which can be built out of two-spin excitations!

3.3 Three-spin excitations

So far we have systematically found the eigenvectors that can be built from states with one and two spins excited from the reference configuration. This has hinged on the appropriate ansatz for the form of these eigenvectors, which in essence constituted a proto-Bethe ansatz in each of their respective cases. However the nature of the Bethe ansatz is more apparent when we consider three-spin excitations. Doing out the three-spin case is a bit of a slog, but it is somewhat illustrative about the general case. Having noticed some patterns in the one and two-spin excitation cases above, we can write down the following conditions for our would-be coefficients for the three-spin excitations that must be satisfied to construct and eigenvector:

$$\begin{aligned}\lambda\psi(n,m,l) = (L-12)\psi(n,m,l) + 2[&\psi(n-1,m,l) + \psi(n+1,m,l) + \\ &\psi(n,m-1,l) + \psi(n,m+1,l) = \psi(n,m,l-1) + \psi(n,m,l+1)]\end{aligned} \tag{19}$$

$$\begin{aligned}\lambda\psi(n,n+1,l) = (L-8)\psi(n,n+1,l) + \\ 2[\psi(n-1,n+1,l) + \psi(n,n+2,l) + \psi(n,n+1,l-1) + \psi(n,n+1,l+1)]\end{aligned} \tag{20}$$

$$\begin{aligned}\lambda\psi(n,m,m+1) = (L-8)\psi(n,m,m+1) + \\ 2[\psi(n-1,m,m+1) + \psi(n+1,m,m+1) + \psi(n,m-1,m+1) + \psi(n,m,m+2)]\end{aligned} \tag{21}$$

$$\psi(n+L, m+L, l+L) = \psi(n,m,l) \tag{22}$$

$$\psi(n,m,L+1) = \psi(1,n,m). \tag{23}$$

The first of these treats the cases where the three excitations are separated along the length of the chain. The second and third handle the two body collisions, and the last two specify periodic boundary conditions. The goal is to make an ansatz for $\psi(n,m,l)$ which allows us to satisfy these conditions. However, first we will point out a very important fact.

3.3.1 Three body collision is redundant

In the above, we have two collision conditions instead of just one as we had in the two-spin excitation case: this is because there are two possible two-body collisions when we have three spin excitations along our chain. However there is a third collision, a three-body collision, that can also occur for states where all three spins are neighboring each other, which corresponds to coefficients of form $\psi(n,n+1,n+2)$. The additional condition for these coefficients is

$$\lambda\psi(n,n+1,n+2) = (L-4)\psi(n,n+1,n+2) + 2\psi(n-1,n+1,n+2) + 2\psi(n,n+1,n+3). \tag{24}$$

This would seem to complicate our problem, as we now have an addition condition to satisfy. However, here something very nice happens. One can see by explicit calculation that the three-body collision condition is automatically satisfied if we have the first three conditions from above satisfied, which are just the generic case and the two body collisions. The way to see this (which the reader should try!) is to plug in the appropriate n, m and l so that one gets an equation for $\psi(n,n+1,n+2)$. For example, in the first equation, one should set $m = n+1, l = n+2$. The fact that the three body collision condition is automatically satisfied by the two body conditions is the key to the solvability (the *integrability*) of this model.

3.3.2 Finding the Bethe equations

Having determined that we do not need the three-body collision condition in order to fully diagonalize our matrix, we will proceed to do without it. For the three-spin excitation case, we make the ansatz

$$\psi(n,m,l) = Az_1^n z_2^m z_3^l + Bz_1^n z_3^m z_2^l + Cz_2^n z_1^m z_3^l + Dz_2^n z_3^m z_1^l + Ez_3^n z_1^m z_2^l + Fz_3^n z_2^m z_1^l. \tag{25}$$

This shows the structure of the Bethe ansatz: it is a sum of $p!$ terms of varying amplitudes, and our job is to find the ratios of the $p!$ amplitudes as well as the complex numbers $z_1, \ldots, z_p$.

Plugging in our ansatz to the three-spin equations, λ can be found to be $\lambda = L - 12 + 2\left(z_1 + z_2 + z_3 + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$ and therefore can be eliminated, and one of the boundary conditions gives $z_1^L z_2^L z_3^L = 1$. Then there are three remaining conditions. These conditions must be satisfied no matter the values of the $\{z_i\}$ and n, m, l. Therefore we can isolate terms of like powers in z_1, z_2, z_3 and set their coefficients to 0. This gives us a number of conditions on the amplitudes $A, \ldots, F$. In particular we find that

$$\frac{A}{C} = -\frac{z_1z_2 - 2z_1 - 1}{z_1z_2 - 2z_2 - 1}, \quad \frac{D}{F} = -\frac{z_2z_3 - 2z_2 + 1}{z_2z_3 - 2z_3 + 1}, \quad \frac{B}{E} = -\frac{z_1z_3 - 2z_1 + 1}{z_1z_3 - 2z_3 + 1}, \tag{26}$$

$$\frac{A}{B} = -\frac{z_2z_3 - 2z_2 + 1}{z_2z_3 - 2z_3 + 1}, \quad \frac{C}{D} = -\frac{z_1z_3 - 2z_1 + 1}{z_1z_3 - 2z_3 + 1}, \quad \frac{E}{F} = -\frac{z_1z_2 - 2z_1 + 1}{z_1z_2 - 2z_2 + 1}. \tag{27}$$

Note that we have found 6 conditions even though just 5 would have been enough to uniquely specify all the amplitudes. This hints at some kind of algebraic structure between the amplitudes. If we start at one amplitude, say A, and find its relationship to, say E, there is more than one way to do this, and these must yield the same answer.

Thus far we have used all the conditions (eigenvalue condition, two collision conditions, and $\psi(n+L,m+L,l+L) = \psi(n,m,l)$) except the second periodic boundary condition, which is $\psi(n,m,L+1) = \psi(1,n,m)$. Enforcing this last condition will allow us to obtain a closed set of equations for the $\{z_i\}$. This last condition reads

$$0 = A(z_1^n z_2^m z_3^{L+1} - z_1 z_2^n z_3^m) + B(z_1^n z_2^{L+1} z_3^m - z_1 z_2^m z_3^n) + C(z_1^m z_2^n z_3^{L+1} - z_1^n z_2 z_3^m) + \tag{28}$$

$$D(z_1^{L+1} z_2^n z_3^m - z_1^m z_2 z_3^n) + E(z_1^m z_2^{L+1} z_3^n - z_1^n z_2^m z_3) + F(z_1^{L+1} z_2^m z_3^n - z_1^m z_2^n z_3). \tag{29}$$

Performing a similar exercise to enforce consistency by setting the coefficients of each monomial to 0, we obtain

$$z_1^L = \frac{B}{F} = \frac{A}{D}, \quad z_2^L = \frac{C}{B} = \frac{D}{E}, \quad z_3^L = \frac{F}{C} = \frac{E}{A}. \tag{30}$$

Given our expressions for the amplitudes above, one can check that these equations are equivalent to the *Bethe equations*

$$z_i^L = \prod_{k=1}^{3} \frac{z_i z_k - 2z_i + 1}{z_i z_k - 2z_k + 1}. \tag{31}$$

As before, we have reduced the task at hand to solving these coupled algebraic equations!

3.4 Multi-spin excitations and the Bethe equations

For general p, the Bethe ansatz is

$$\psi(n_1, \dots n_p) = \sum_{\sigma \in S_p} A_\sigma \prod_{k=1}^{p} z_{\sigma(k)}^{n_k}. \tag{32}$$

We have already seen that the three body collision factors into two body collisions and thus provides no new information. This property is what makes our system tractable. In fact collisions of any degree will factor totally into two-body collisions, and so these are the only conditions that we need to enforce.

Therefore the number of conditions we can enforce in order to solve for our eigenfunctions and eigenvectors is one eigenvalue condition, $p-1$ two-body collisions, and two boundary conditions, namely from translating all excitations by L and from the case where the last excitation is at the lattice site L. This is a total of $p+1$ conditions. The number of unknowns in our ansatz is $p! - 1 + p$, for the amplitudes of the various terms (modulo an overall factor) and and the values of the z_k themselves. However, since the amplitudes are related to one another by functions of the z_k, we need only to solve for one of the ratios, and the others will follow. Note that the reduction in complexity due to the factorization of the collisions is commensurate with the reduction in complexity from the algebraic relationship between amplitude ratios. Therefore the total number of unknowns is also $p+1$. This will allow us to solve for the z_i for $i \in \{1, \dots, p\}$. In general one finds that these satisfy

$$z_i^L = (-1)^{p+1} \prod_{k=1}^{p} \frac{z_i z_k - 2z_i + 1}{z_i z_k - 2z_k + 1}, \tag{33}$$

for excitations of p spins. These are the general Bethe equations.

The fact that the number of conditions and the number of unknowns are both $p+1$ is highly nontrivial and is what makes this model solvable by the Bethe ansatz. In general this will not be true: but for a class of so-called *integrable* models this property holds, and allows for solution by the Bethe ansatz.

4 Solving the Bethe equations

In principle, as we have said, the full spectrum of our Hamiltonian can be found by solving the Bethe equations for p from 1 to L. Although we have seriously simplified the problem from the original diagonalization of a $2^L \times 2^L$ matrix to the solution of p coupled algebraic equations, finding the solutions of these equations is still quite nontrivial.

To see how this works, consider the case of $p = 2$. Then the Bethe equations become the same for z_1 and z_2, namely

$$(1^{1/L} + 1)(z^L + 1) - 2(z + 1^{1/L} z^{L-1}) = 0, \tag{34}$$

where we have used the fact that $z_1 z_2 = 1^{1/L}$ where $1^{1/L}$ is an Lth root of unity. This equation has L solutions which can be found numerically. Furthermore, solutions where $z_1 = z_2$ are invalid, as can be checked given the form of the Bethe ansatz for $p = 2$ above. Therefore we have $\binom{L}{2}$ solutions for the pair z_1 and z_2. The same idea continues for larger values of p, though it is much more difficult (even numerically) and I do not know of a systematic way of solving these equations. I believe that for each p, we get $\binom{L}{p}$ valid solutions of the $\{z_1, \dots z_p\}$, which therefore account for all 2^L eigenvalues since $\sum_{p=1}^{L} \binom{L}{p} = 2^L$. There are related cases (which we will discuss in other notes) where the Bethe equations are easier to solve, but the XXX chain requires quite a lot of work to solve.

We will end our discussion here for now: having discussed the derivation of the Bethe equations and the nontrivial facts that allow us to solve the XXX model by the appropriate Bethe ansatz. There is lots more to be said about solving the Bethe equations, as well as different incarnations of the Bethe ansatz (algebraic, functional, etc.) which we may address in further notes.

5 References

A nice lecture by Kirone Mallick on the Bethe ansatz in a different context
Bethe's original paper, translated into English
Introduction to Bethe Ansatz I, Karbach & Müller 1998